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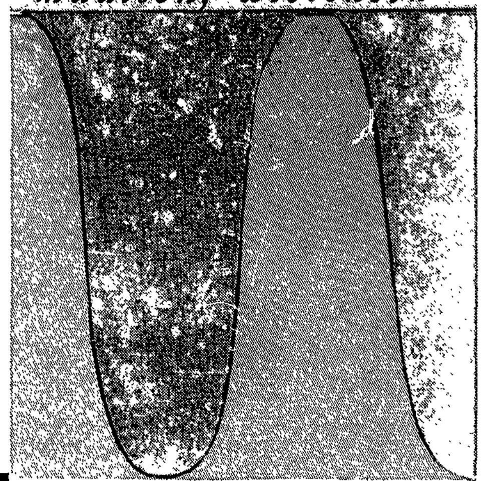

POWER SERIES WHOSE SECTIONS HAVE
ZEROS OF LARGE MODULUS

J. D. Buckholtz

MRC Technical Summary Report #398
April 1963

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ABSTRACT

Given a power series, $\sum a_p z^p$, let r_n denote the smallest modulus of a zero of $s_n(z) = \sum_{p=0}^{n} a_p z^p$, $n = 1, 2, 3, \ldots$. Upper and lower estimates for r_n are obtained under the hypothesis that $\sum a_p z^p$ is the power series for an entire function without zeros. For certain classes of such series, asymptotic formulas for r_n are derived. Characterizations (in terms of r_n) are obtained for entire functions of the forms $\exp\{P(z)\}$ and $\exp\{g(z)\}$, where $P(z)$ is a polynomial and $g(z)$ is an entire function of finite order.

POWER SERIES WHOSE SECTIONS HAVE ZEROS OF LARGE MODULUS

J. D. Buckholtz

1. Introduction. Several theorems in the theory of polynomials deal with the problem of obtaining bounds for the modulus of one or more zeros of a polynomial, $a_0 + a_1 z + \dots + a_n z^n$, when certain of the coefficients, $a_0, a_1, \dots a_k$, are regarded as fixed, and the remaining are arbitrary (cf. [3, ch. 8]). In the present paper we apply results of this nature to partial sums of a power series $\sum a_p z^p$.

For each positive integer n, r_n will denote the radius of the largest circle with center at $z = 0$ whose interior contains no zero of the nth partial sum,

$$s_n(z) = \sum_{p=0}^{n} a_p z^p .$$

We shall be concerned primarily with growth properties of the sequence $\{r_n\}$. The most interesting case is that in which $\sum a_p z^p$ is the power series for an entire function which omits the value zero. It is not hard to show that this is equivalent to having $\lim r_n = \infty$; one can, however, construct other power series for which $\limsup r_n = \infty$.

Since nothing is lost by doing so, we shall always suppose that $a_0 = 1$. This assumption will be used freely and without explicit mention. For notational

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convenience $\sum$ and $\sum'$ will denote sums taken over the nonnegative and positive integers respectively. When there is no possibility of ambiguity, "lim" will denote a limit taken as the variable becomes infinite.

In §2, upper bounds for r_n are obtained from algebraic relations between the zeros of $s_n(z)$ and the "first few" of the numbers $a_1, a_2, a_3, \ldots$. From algebraic considerations alone, we show that

$$r_n = n^{o(1)} \quad , \tag{1.1}$$

except possibly for certain "exceptional" power series, and, further, that these exceptions must be power series of the form $\exp\{P(z)\}$, where $P(z)$ is a polynomial.

In §3 we use analytic methods to obtain lower bounds for r_n in case $\sum a_p z^p$ is an entire function without zeros. We are able to show that the "apparent exceptions" to (1.1) are actual exceptions, and thus characterize entire functions of the form $\exp\{P(z)\}$ for $P(z)$ a polynomial.

Taken together, the upper and lower bounds yield a number of asymptotic properties of the sequence $\{r_n\}$. For $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of order ρ, we show that

$$\limsup \frac{\log\log n}{\log r_n} = \rho \ .$$

This and similar results are discussed §4 .

In §5 we prove that

$$\limsup r_n \, |a_n|^{1/n} = 1 \quad , \tag{1.2}$$

provided $\sum a_p z^p$ is an entire function of infinite order without zeros. Using (1.2) and the observation that $|a_n|^{-1/n}$ is the geometric mean of the moduli of zeros of $s_n(z)$, we deduce the following: *if* $\sum a_p z^p$ *is an entire function of infinite order without zeros, and* ϵ *and* ϵ' *are positive numbers, then, for infinitely many integers* n, *fewer than* $n\epsilon'$ *zeros of* $s_n(z)$ *have moduli greater than* $r_n(1+\epsilon)$. This result is of some interest in connection with theorems of F. Carlson [2] and P. C. Rosenbloom [7] on zeros of sections of entire series of infinite order.

2. Upper bounds for r_n. Let $\sum' b_p z^p$ be the power series obtained formally from the identity

$$\frac{\sum pa_p z^{p-1}}{\sum a_p z^p} = \sum' pb_p z^{p-1} \quad .$$

Theorem 2.1. <u>If</u> k <u>is a positive integer such that</u> $b_k \neq 0$, <u>and</u> $n \geq k$, <u>then</u> $s_n(z)$ <u>has a zero in the disc</u>

$$|z| \leq \left\{ \frac{n}{k|b_k|} \right\}^{1/k} \quad .$$

Proof. If one lets

$$\frac{s_n'(z)}{s_n(z)} = \sum' p\, b_p^{(n)} z^{p-1}$$

and observes that $b_p^{(n)} = b_p$ for $p \leq n$, the result then follows from a theorem of G. Sz. Nagy [4, 5, and 3, p. 43, ex. 2].

As a consequence of the above, we have

$$r_n = O(n^{1/k}) \tag{2.1}$$

for every value of k for which $b_k \neq 0$. If $\sum a_p z^p = \exp\{P(z)\}$ for some polynomial $P(z)$, then (2.1) holds with k equal to the degree of $P(z)$. If $\sum a_p z^p$ is not a power series of this form, then (2.1) holds for infinitely many k, and we have

$$r_n = n^{o(1)} \quad . \tag{2.2}$$

Corollary 2.2 .

$$\liminf \frac{\log r_n}{\log\log n} \leq \liminf \frac{\log (1/|b_k|)}{k \log k} \quad . \tag{2.3}$$

Proof. From theorem 2.1 we have

$$\frac{\log r_n}{\log k} \leq \frac{\log n}{k \log k} - \frac{1}{k} + \frac{\log(1/|b_k|)}{k \log k} \quad .$$

Choose $n = n(k)$ so that $\log n \sim k$ and let $k \to \infty$.

The above result is of particular interest if $\sum' b_p z^p = g(z)$, where $g(z)$ is an entire function of order ρ . We then have $\sum a_p z^p = \exp\{g(z)\}$, and

$$\limsup \frac{\log\log n}{\log r_n} \geq \rho \ , \tag{2.4}$$

since the right hand side of (2.3) is $1/\rho$.

For certain entire functions $g(z)$, Theorem 2.1 can be used to obtain an extremely good upper bound for r_n . For this purpose we make use of the maximum term function $\mu(r) = \mu(r, g)$ defined by

$$\mu(r) = \max_p |b_p| r^p \ , \tag{2.5}$$

and the central index , $\nu(r)$, which is the largest integer m such that

$$\mu(r) = |b_m| r^m \quad .$$

Theorem 2.3. *Let* $\sum a_p z^p = \exp\{g(z)\}$, *where* $g(z) = \sum' b_p z^p$ *is an entire function of finite order. For each* n, *let* β_n *be the positive number such that* $\mu(\beta_n) = n$, *where* $\mu(r)$ *is defined by* (2.5). *Then for all sufficiently large* n, $s_n(z)$ *has a zero in the disc* $|z| \leq \beta_n$.

Proof. From Theorem 2.1,

$$r_n \leq \left\{\frac{n}{k|b_k|}\right\}^{1/k} \leq \left\{\frac{n}{|b_k|}\right\}^{1/k} .$$

Let $k = \nu(\beta_n)$. Then $|b_k|\,\beta_n^k = \mu(\beta_n) = n$. Therefore $r_n \leq \beta_n$.

It remains to show that $k \leq n$, or equivalently, that $\nu(\beta_n) \leq \mu(\beta_n)$. This is true provided the inequality

$$\nu(r) < \mu(r) \tag{2.6}$$

holds for all suffiently large r. A proof of (2.6) follows easily from the relation [11, p. 34]

$$\limsup \frac{\log \nu(r)}{\log r} = \rho ,$$

where ρ is the order of $g(z)$. The hypothesis that ρ is finite can, consequently, be replaced by (2.6).

3. Lower bounds for r_n. We obtain lower bounds for the numbers r_n under the assumption that $\sum a_p z^p = \exp\{g(z)\}$, where $g(z) = \sum' b_p z^p$ is an entire function. We shall use $G(z)$ to denote the majorant of $g(z)$ defined by

$$G(z) = \sum' |b_p| z^p \quad . \tag{3.1}$$

We note for future use that the order of $G(z)$ is the same as that of $g(z)$; we shall also need the inequality

$$\sum |a_p| r^p \le \exp\{G(r)\} \quad \underline{\text{if}} \quad r \ge 0 \quad . \tag{3.2}$$

A proof of (3.2) follows from expanding $\exp\{G(r)\}$ as a power series in r and observing that the coefficient of r^p is at least as great as $|a_p|$.

Theorem 3.1. *Let $\sum a_p a^p = \exp\{g(z)\}$, where $g(z)$ is an entire function with majorant $G(z)$ defined by (3.1). If n is a positive integer, then*

$$r_n > r \exp\{-\frac{2G(r)}{n}\} \quad \underline{\text{for all}} \quad r \ge 0 \quad . \tag{3.3}$$

In particular, if α_n is the positive number such that $G(\alpha_n) = n$, then

$$r_n > \frac{\alpha_n}{e^2} \quad . \tag{3.4}$$

Furthermore, if $g(z)$ is not a polynomial, then for $\epsilon > 0$ one has

$$r_n > \alpha_n(1 - \epsilon) \tag{3.5}$$

for all sufficiently large n.

Proof. Suppose $r > 0$ and let $f(z) = \sum a_p z^p$. We shall establish (3.3) by showing that, if $|z| \le r \exp\{-2G(r)/n\}$, then

$$\left| 1 - \frac{s_n(z)}{f(z)} \right| < 1 \quad ,$$

and therefore $s_n(z) \neq 0$. This is obvious if $z = 0$; if $0 < |z| \le r \exp\{-2G(r)/n\}$, then $0 < |z| < r$, and

$$\frac{1}{|f(z)|} = |e^{-g(z)}| < \exp\{G(r)\} \; .$$

Also,

$$\begin{aligned} |f(z) - s_n(z)| &= \Big|\sum_{p=n+1}^{\infty} a_p z^p\Big| \\ &\le \left|\frac{z}{r}\right|^n \sum_{p=n+1}^{\infty} |a_p| r^p \left|\frac{z}{r}\right|^{p-n} \\ &< \left|\frac{z}{r}\right| \sum_{p=n+1}^{\infty} |a_p| r^p \\ &< \left|\frac{z}{r}\right|^n \sum |a_p| r^p \\ &\le \left|\frac{z}{r}\right|^n \exp\{G(r)\} \end{aligned}$$

by virtue of (3.2). Therefore

$$\begin{aligned} \left| 1 - \frac{s_n(z)}{f(z)} \right| &= \left| \frac{f(z) - s_n(z)}{f(z)} \right| \\ &< \left[\left|\frac{z}{r}\right| \exp\left\{\frac{2G(r)}{n}\right\} \right]^n \\ &\le 1 \; , \end{aligned}$$

since $|z| \leq r \exp\{-2G(r)/n\}$. This proves (3.3) . If $r = \alpha_n$, we have (3.4) .

The proof of (3.5) depends on the following property of $G(r)$ [6, vol. 2, p. 4]: _if_ $g(z)$ _is not a polynomial and_ $0 < c < 1$, _then_

(3.6) $$\lim_{r \to \infty} \frac{G(cr)}{G(r)} = 0 \ .$$

We now make use of (3.6) and the sequence $\{\alpha_n\}$ to construct a sequence $\{c_n\}$ such that

$$\lim c_n = 1 \text{ and } \lim \frac{G(c_n \alpha_n)}{G(\alpha_n)} \doteq 0 \ .$$

If in (3.3) we let $r = c_n \alpha_n$, we have

$$r_n > c_n \alpha_n \exp\{-\frac{2G(c_n \alpha_n)}{n}\}$$

$$= c_n \alpha_n \exp\{-\frac{2G(c_n \alpha_n)}{G(\alpha_n)}\} \ .$$

Since

$$\lim c_n \exp\{-\frac{2G(c_n \alpha_n)}{G(\alpha_n)}\} = 1 \quad ,$$

this proves (3.5) .

4. Asymptotic properties of the sequence $\{r_n\}$. In a number of cases, fairly precise information about the sequence $\{r_n\}$ can be obtained by comparing the upper bounds of §2 with the lower bounds developed in §3 .

Theorem 4.1. *If* $\sum a_p z^p = \exp\{P(z)\}$, *where* $P(z)$ *is a polynomial of degree* k, *then there are positive numbers* A *and* B *such that*

$$An^{1/k} < r_n < Bn^{1/k}, \quad n = 1, 2, 3, \ldots .$$

The proof, which is omitted, follows easily from (2.1) and (3.4) .

In view of (2.2), one sees that Theorem 4.1 characterizes power series for entire functions of the form $\exp\{P(z)\}$. Among all power series, the exponential series (more accurately the series for ae^{bz}) is the only one for which r_n increases as rapidly as a linear function. Zeros of sections and remainders of this series have been investigated by G. Szegö [9].

Theorem 4.2 is to some extent analogous to a theorem of M. Tsuji [10] on the maximum modulus of zeros of sections of an entire series.

Theorem 4.2. *If* $\sum a_p z^p = \exp\{g(z)\}$, *where* $g(z)$ *is an entire function of order* ρ $(0 \leq \rho \leq \infty)$, *then*

$$\limsup \frac{\log\log n}{\log r_n} = \rho .$$

Proof. Since the order of $G(z)$ is also ρ, we have

$$\rho = \limsup_{r\to\infty} \frac{\log\log G(r)}{\log r} \geq \limsup_{n\to\infty} \frac{\log\log n}{\log \alpha_n}$$

$$\geq \limsup \frac{\log\log n}{\log(r_n e^2)}$$

by (3.4) . Since $\log(r_n e^2) \sim \log r_n$,

$$\rho \geq \limsup \frac{\log\log n}{\log r_n} .$$

This, together with (2.4), completes the proof.

If in Theorem 4.2 $0 < \rho < \infty$ and $g(z)$ is of type τ $(0 \leq \tau \leq \infty)$, one can prove a sharper result, namely, that

$$\limsup \frac{\log n}{r_n^{\rho}} = \tau .$$

For this one needs (3.5) in place of (3.4); the "$\geq$" half of the result is obtained from Theorem 2.1 by a procedure similar to the proof of Corollary 2.2. In this case one chooses $n = n(k)$ so that $\log n \sim k/\rho$.

If $g(z)$ is of finite order, the asymptotic relation [6, vol. 2, p. 8]

$$\log G(r) \sim \log \mu(r, G) = \log \mu(r, g) \tag{4.1}$$

can be used to obtain information about the relative sizes of α_n and β_n .

Theorem 4.3. _If_ $\sum a_p z^p = \exp\{g(z)\}$, _where_ $g(z)$ _is an entire function of finite order, then_

$$\log \alpha_n \sim \log r_n \sim \log \beta_n .$$

Proof. Since $\alpha_n e^{-2} < r_n \leq \beta_n$ for large n, it suffices to prove that $\log \alpha_n \sim \log \beta_n$. Using (4.1), we have

$$\log G(\beta_n) \sim \log \mu(\beta_n) = \log n = \log G(\alpha_n) ,$$

so that

$$\text{(4.2)} \qquad \log G(\alpha_n) \sim \log G(\beta_n) \quad .$$

Since $\log G(r)$ is an increasing and convex function of $\log r$, one can conclude that $\log \alpha_n \sim \log \beta_n$. This completes the proof.

Convexity of $\log G(\exp(r))$ is considerably more than one needs for the proof of Theorem 4.3. We take advantage of this to prove that $\alpha_n \sim r_n \sim \beta_n$ under an added hypothesis that $G(r)$ satisfies a relatively weak growth condition.

Theorem 4.4. _Let_ $\sum a_p z^p = \exp\{g(z)\}$, _where_ $g(z)$ _is an entire function of finite order. Suppose in addition that_ $\log G(r) \sim r^\delta H(r)$, _where_ $\delta > 0$ _and_ $H(r)$ _is nondecreasing. Then_ $\alpha_n \sim r_n \sim \beta_n$.

Proof. Suppose $\epsilon > 0$. The condition on $G(r)$ guarantees that $g(z)$ is not a polynomial; therefore $\alpha_n(1-\epsilon) < r_n \leq \beta_n$ for sufficiently large n, and we need only to prove that $\alpha_n \sim \beta_n$.

From (4.2) and the condition on $G(r)$, we have $(\alpha_n/\beta_n)^\delta \sim H(\beta_n)/H(\alpha_n)$. But $(\alpha_n/\beta_n)^\delta \leq 1$, and $H(\beta_n)/H(\alpha_n) \geq 1$ since $H(r)$ is nondecreasing. Therefore $\alpha_n \sim \beta_n$.

As a special case of the above, we note that the condition

$$\log G(r) \sim \tau r^\rho$$

for positive numbers ρ and τ implies that

$$r_n \sim \left\{\frac{\log n}{\tau}\right\}^{1/\rho} \quad .$$

5. A relation between $\{r_n\}$ and $\{a_n\}$. In this section we restrict our attention to the case $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function which is not a polynomial. This is equivalent to requiring that $\sum a_p z^p$ be an entire function of infinite order without zeros. We shall compare the lower bound (3.5) for r_n with the "obvious" upper bound,

$$r_n \leq \frac{1}{|a_n|^{1/n}} \quad \underline{\text{if}} \quad a_n \neq 0 \ . \tag{5.1}$$

(The right hand side of (5.1) is the geometric mean of the moduli of zeros of $s_n(z)$.) Our principal result is the following:

Theorem 5.1. *If $\sum a_p z^p$ is an entire function of infinite order without zeros, then*

$$\limsup \alpha_n |a_n|^{1/n} = \limsup r_n |a_n|^{1/n} = 1 \ .$$

Before proving Theorem 5.1 we shall consider two of its corollaries.

Corollary 5.2. *Let $\sum a_p z^p$ satisfy the hypothesis of Theorem* 5.1. *If* $\epsilon > 0$, *then*

$$\alpha_n(1-\epsilon) < r_n \quad \text{and} \quad r_n |a_n|^{1/n} \leq 1$$

for all sufficiently large n, *and*

$$\alpha_n(1+\epsilon) > r_n > \frac{1-\epsilon}{|a_n|^{1/n}}$$

for infinitely many n .

Proof. Theorem 5.1 and inequalities (3.5) and (5.1).

One sees, therefore, that (3.5) and (5.1) are, in a sense, "best possible" results.

Corollary 5.3. *Let* $\sum a_p z^p$ *satisfy the hypothesis of Theorem* 5.1. *If* $\epsilon > 0$ *and* $\epsilon' > 0$, *then for infinitely many integers* n, *fewer than* $n\epsilon'$ *zeros of* $s_n(z)$ *have moduli greater than* $r_n(1+\epsilon)$.

Proof. Choose δ so that

$$0 < \delta < 1 - (1+\epsilon)^{-\epsilon'} .$$

If n is a positive integer for which

$$r_n > \frac{1-\delta}{|a_n|^{1/n}} ,$$

then an easy calculation shows that fewer than $n\epsilon'$ zeros of $s_n(z)$ have moduli larger than $r_n(1+\epsilon)$.

Proof of Theorem 5.1. From (3.5) and (5.1) it follows that

$$\limsup \alpha_n |a_n|^{1/n} \leq \limsup r_n |a_n|^{1/n} \leq 1 ;$$

consequently we have only to prove that

$$\limsup \alpha_n |a_n|^{1/n} \geq 1 . \tag{5.2}$$

For clarity, the proof of (5.2) will be divided into three lemmas. In the first two, the hypothesis of Theorem 5.1 is presupposed.

Lemma 5.1a. *For all* $r > 0$,

$$\alpha_n > r \exp\{-\frac{2G(r)}{n}\} . \tag{5.3}$$

Proof. The function $r \exp\{-2G(r)/n\}$ assumes its maximum at the number $r = \gamma_n$ such that $2\gamma_n G'(\gamma_n) = n$. Since $2r G'(r) > G(r)$ for all $r > 0$, we have

$$\alpha_n > \gamma_n > \gamma_n \exp\{-\frac{2G(\gamma_n)}{n}\} .$$

Lemma 5.1b. *If* $\epsilon > 0$, *then*

$$\alpha_n > \frac{r}{1+\epsilon} [M(r)]^{-8/n\epsilon} \text{ for all } r > 0, \text{ where}$$

$$M(r) = \max_{|z|=r} |\sum a_p z^p| . \tag{5.4}$$

Proof. The proof depends on the following variant of the Borel-Carathéodory inequality (proved, but not explicitly stated, in [11, pp. 17-20]):

If $0 < r < R$ *and* $\sum' b_p z^p$ *is an entire function, then*

$$\sum' |b_p| r^p \leq \frac{4r}{R-r} A(R) , \tag{5.5}$$

where

$$A(R) = \max_{|z|=R} \{\text{Re} \sum' b_p z^p\} .$$

If we rewrite (5.3) in the form

$$\alpha_n > \frac{r}{1+\epsilon} \exp\{-\frac{2}{n} G(\frac{r}{1+\epsilon})\} ,$$

and in (5.5) replace r and R by $r/(1+\epsilon)$ and r respectively, we have

$$\alpha_n > \frac{r}{1+\epsilon} \exp\{-\frac{8}{n\epsilon} A(r)\} .$$

Since $A(r) = \log M(r)$, the result follows.

Our third lemma is of a more general nature and applies to all entire functions $\sum a_p z^p$ of infinite order. (The condition $a_0 = 1$ is still presupposed, although the result is true without it.)

Lemma 5.1c. *Suppose that* $\sum a_p z^p$ *is an entire function of infinite order and* $K \geq 1$. *Let*

$$u_n = u_n(K) = \min_{r>0} \frac{[M(r)]^K}{r^n} , \qquad n = 1, 2, 3, \ldots ,$$

where $M(r)$ *is given by* (5.4) . *Then*

$$\limsup_{n\to\infty} \frac{|a_n|^{1/n}}{|u_n|^{1/n}} = 1 .$$

Proof. For all $r > 0$,

$$|a_n| \leq \frac{M(r)}{r^n} \leq \frac{[M(r)]^K}{r^n}$$

by the Cauchy inequality. Therefore $|a_n| \leq u_n$, and the proof will be complete if we show that

$$\limsup \frac{|a_n|^{1/n}}{u_n^{1/n}} \geq 1 \quad . \tag{5.6}$$

To establish (5.6), we observe that

$$M(r) \leq \sum |a_p| r^p \leq \sum \Psi_p r^p \quad ,$$

where (n, Ψ_n), $n = 0, 1, 2, \ldots$ are points on the Newton polygon associated with $\sum a_p z^p$ (cf. [11, ch. 2] and [6, vol. 2, ch. 1]). Let $d_0 = 1$ and $d_n = \Psi_{n-1}/\Psi_n$, $n = 1, 2, 3, \ldots$. The sequence $\{d_n\}$ is nondecreasing and has limit ∞ ; therefore the function $\Psi(z) = \sum \Psi_p z^p$ is, in the terminology of [1], a <u>comparison function.</u> If we let

$$r = r' = \frac{n}{n+1} d_n \quad ,$$

we have [1, p. 7]

$$M(r') \leq \Psi(r') \leq (n+1) d_n^n \Psi_n \quad .$$

A short computation shows that

$$u_n^{1/n} \leq \frac{X_n Y_n}{d_n} \quad ; \tag{5.7}$$

where

$$X_n = (1 + 1/n)(n+1)^{K/n} \quad ,$$

and

$$Y_n = (d_n \Psi_n^{1/n})^K \quad .$$

Let N denote the set of integers n for which $|a_n| = \Psi_n$. Since $\sum a_p z^p$ is of infinite order, it follows from [2, p. 4, eq. 9] that there is an infinite subset N_0 of N such that

$$\lim_{n \in N_0} d_n \Psi_n^{1/n} = \lim_{n \in N_0} d_n |a_n|^{1/n} = 1 \quad . \tag{5.8}$$

Equation (5.8) is, in fact, equivalent to

$$\liminf \frac{\log \mu(r)}{\nu(r)} = 0 ,$$

where in this case, $\mu(r)$ and $\nu(r)$ denote the maximum term and central index of the series $\sum a_p z^p$. With this formulation, (5.8) also follows as a special case of a theorem of S. M. Shah [8].

Making use of (5.7) and (5.8), we have

$$\limsup \frac{|a_n|^{1/n}}{u_n^{1/n}} \geq \limsup \frac{d_n |a_n|^{1/n}}{X_n Y_n}$$

$$\geq \lim_{n \in N_0} \frac{d_n |a_n|^{1/n}}{X_n Y_n}$$

$$= 1 .$$

This completes the proof of Lemma 5.1c.

It is now an easy matter to complete the proof of Theorem 5.1. In Lemma 5.1b choose $0 < \epsilon < 8$. In Lemma 5.1c let $K = 8/\epsilon$ and choose r so that

$$\frac{[M(r)]^{8/\epsilon}}{r^n} = u_n .$$

Then

$$\alpha_n |a_n|^{1/n} \geq \frac{|a_n|^{1/n}}{1+\epsilon} r[M(r)]^{-8/n\epsilon} = \frac{1}{1+\epsilon} \left|\frac{a_n}{u_n}\right|^{1/n} .$$

Therefore $$\limsup \alpha_n |a_n|^{1/n} \geq \frac{1}{1+\epsilon} \quad ,$$

and, since ϵ is arbitrary, (5.2) follows. This completes the proof.

Theorem 5.1 adds an interesting footnote to certain more general results on zeros of sections of power series. If $\sum a_p z^p$ is an arbitrary entire function of infinite order and $\epsilon > 0$, it is known [2, 7] that for all sufficiently large integers n in the set N_0 (of Lemma 5.1c), all but $o(n)$ zeros of $s_n(z)$ lie in the annulus

$$|a_n|^{-1/n}(1-\epsilon) < |z| < |a_n|^{-1/n}(1+\epsilon) \quad .$$

If $\sum a_p z^p$ omits the value zero, it follows from Theorem 5.1 that for all sufficiently large $n \in N_0$, no zero of $s_n(z)$ lies in the interior of the inner circle of the annulus.

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